Comment on "π kinks in strongly ac driven sine-Gordon systems"

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V. Zharnitsky, I. Mitkov, and N. Gronbech-Jensen [Phys. Rev. E **58**, 1, R52 (1998)] found that π kinks can propagate in strongly perturbed, directly driven rescaled sine-Gordon system provided that the parameters are chosen to make 2π kink localization vanish. In this paper we would like to note that beside π and 2π kinks there can exist other kinklike solutions due to the fact that two unstable equilibria in the sine-Gordon phase emerging at a critical value of the drive amplitude are not necessarily separated by π, to the contrary with the result of Zharnitsky, Mitkov, and Gronbech-Jensen. As a result, for the nondissipative system two one-parameter families of kink solutions exist that in the degenerate case become a one-parameter family of π-kink solutions obtained in Zharnitsky, Mitkov, and Gronbech-Jensen. In the dissipative case velocity is selected for each of the two families of kink solutions by the balance between perturbations.

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In a recent paper Zharnitsky *et al.* [1] found for nondissipative ac driven sine-Gordon equation (SGE) a one-parameter family of π kinks moving with any prescribed velocity. In the case of damped and driven SGE it was shown that velocity is selected and only one of the π kinks survives. The purpose of this Comment is to show that other kinklike solutions can be easily obtained using the elegant techniques of Zharnitsky *et al.*, if one eliminates small inaccuracies in [1].

Following [1], consider firstly the equation of motion for a directly forced pendulum $\ddot{\phi}+\sin\phi = Mf(\omega t)$, where f is a mean-zero periodic function, M is a constant, t represents a normalized time, and $\omega \gg 1$. After applying the transformation $\phi=\theta+M\omega^{-2}F(\omega t)$ [where $F''(\tau)=f(\tau)$], one obtains the equation

$$\ddot{\theta}+\sin[\theta+M\omega^{-2}F(\omega t)]=0 \tag{1}$$

with the corresponding Hamiltonian $H(p,\theta)$. Invoking series of canonical transformations described in [1], one finally gets the Hamiltonian $H_1=\tilde{H}(P,\Theta)+O(\omega^{-3})$, where

$$\tilde{H}(P,\Theta)=\frac{P^2}{2}-C\cos(\Theta-\gamma)-\frac{\omega^{-2}}{2}D\cos(2\Theta-\delta), \tag{2}$$

here P and Θ are new canonical variables and C, D, γ, δ are constants depending on perturbation $Mf(\omega t)$ [1]. The simple but important question is the existence and location of equilibria of the system [Eq. (2)].

In [1], it is said that for $C\neq 0$ there is only one stable equilibrium $\Theta=\gamma$ and one unstable equilibrium $\Theta=\pi+\gamma$, for large frequencies; as C passes through 0, a bifurcation occurs and (for $C=0$) the system has two stable equlibria given by $\Theta=\delta/2$, $\pi+\delta/2$ and two unstable equilibria given by $\Theta=\pi/2+\delta/2$, $3\pi/2+\delta/2$. This is slightly inaccurate. Althouth it is true that for any $C\neq 0$ there are frequencies large enough for the system to have only one unstable and one stable equilibrium, it is obvious that for any frequency, we can find $C\neq 0$ such that the system would have two stable and two unstable equilibria. This is the key point that leads to the existence of new kink solutions in the forced SGE. Denote $\kappa=C\omega^2/2D$ and let $D>0$. One can see that in case $2\gamma=\delta-\pi$ and $\kappa<1$ there are two stable equilibria given by $\Theta_{a,b}=\gamma+\arccos\kappa, \gamma-\arccos\kappa$ and two unstable equlibria given by $\Theta_{c,d}=\gamma, \pi+\gamma$. The distance between two stable equlibria is equal to $\beta=2\arccos\kappa$ and becomes equal to π for $C=0$. It is important that these equlibria are on the same level of the Hamiltonian, i.e., $\tilde{H}(0,\Theta_a)=\tilde{H}(0,\Theta_b)$. We are interested in the stable equlibria of $\tilde{H}(P,\Theta)$ because in the consideration of the forced SGE the Hamiltonian $\hat{H}(P,\Theta)=P^2/2+C\cos(\Theta-\gamma)+(\omega^{-2}/2)D\cos(2\Theta-\delta)$ will arise. The Hamiltonian $\hat{H}(P,\Theta)$ has the unstable equilibria coincident with the stable equilibria of $\tilde{H}(P,\Theta)$ (see Fig. 1). From now on, we assume that $2\gamma=\delta-\pi$ (we will show below that this condition is automatically fulfilled when the perturbation is sinusoidal).

Solving the equations of motion corresponding to Hamiltonian $\hat{H}(P,\Theta)$, one can get the following separatrix solutions:

$$\Theta_1(t)=\gamma+2\arctan\left[\sqrt{\frac{1-\kappa}{1+\kappa}}\tanh\left(\sqrt{\frac{D(1-\kappa^2)}{2\omega^2}}t\right)\right], \tag{3}$$

$$\Theta_2(t)=\gamma+\pi+2\arctan\left[\sqrt{\frac{1+\kappa}{1-\kappa}}\tanh\left(\sqrt{\frac{D(1-\kappa^2)}{2\omega^2}}t\right)\right],$$

where $\Theta_1(t)$ is the separatrix solution with the initial condition $\Theta_1(0)=\gamma$, and $\Theta_2(t)$ is the separatrix solution with $\Theta_2(0)=\pi+\gamma$.

Consider now the damped and driven SGE:

$$\phi_{tt}-\phi_{xx}+\sin\phi=Mf(\omega t)-\alpha\phi_t+\eta. \tag{4}$$

Following [1], introduce a new phase $\theta=\phi-G(t)$, where $\ddot{G}+\alpha\dot{G}=Mf(\omega t)$. The new equations of motions are

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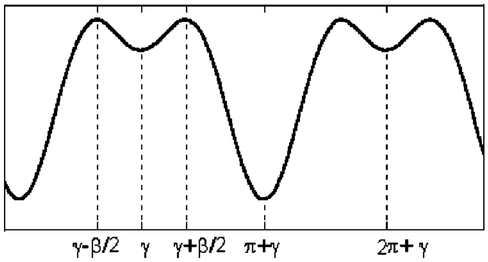

FIG. 1. The effective potential $\hat{H}(0,\Theta)=C\cos(\theta-\gamma)+(\omega^{-2}/2)D\cos(2\Theta-\delta)$ in the case $2\gamma=\delta-\pi$ and $\kappa=C\omega^2/2D<1$.

$$\theta_t=p,\quad p_t=\theta_{xx}-\alpha p+\eta-\sin[\theta+G(\omega t)]. \tag{5}$$

Applying a series of transformations described in [1] to Eq. (5) and neglecting terms $\sim\omega^{-3}$ one gets the following system of equations [compare with Eq. (16) of [1]]:

$$\Theta_t=P,$$
$$P_t=\Theta_{xx}-\alpha P+\eta-C\sin(\Theta-\gamma)+\frac{D}{\omega^2}\sin(2\Theta-2\gamma). \tag{6}$$

Let us substitute a traveling wave ansatz, $Z=x-ct$ in Eq. (6). In the zeroth order in α and η, we get the equation $\ddot{\Theta}=\sin(\Theta-\gamma)-\omega^{-2}D\sin(2\Theta-2\gamma)$, where overdots denote differentiating over new time $Z/\sqrt{1-c^2}$. The Hamiltonian of the latter equation is $\hat{H}(P,\Theta)$ and its separatrix solutions are $\Theta_{1,2}(Z/\sqrt{1-c^2})$. These solutions are kinks whose heights [i.e., $\Theta(+\infty)-\Theta(-\infty)$] are equal to β and $2\pi-\beta$, respectively. In the degenerate case $\beta=2\arccos\kappa=\pi$ these two one-parameter families of kink solutions give us π kinks obtained in [1] (note that $\arctan[\tanh(x/2)]\equiv\arctan[\exp(x)]-\pi/4$). When $\alpha\neq 0$ and $\eta\neq 0$, only two of these solutions are selected out because of the energy balance consideration [1]. Each of the one-parameter families of kink solutions $\Theta_1(Z/\sqrt{1-c^2})$ and $\Theta_2(Z/\sqrt{1-c^2})$ gives one solution, with the velocities c_1 and c_2 correspondingly, where

$$c_{1,2}=\frac{-\eta}{\alpha}\,\frac{\int_{-\infty}^{+\infty}\Theta'_{1,2}\,dZ}{\int_{-\infty}^{+\infty}(\Theta'_{1,2})^2\,dZ}=-\frac{\sigma_{1,2}\eta\omega}{\sqrt{8\alpha^2D+\eta^2\omega^2\sigma_{1,2}^2}}. \tag{7}$$

Here we denoted

$$\sigma_{1,2}=\frac{4\arctan\sqrt{\dfrac{1\mp\kappa}{1\pm\kappa}}}{\sqrt{1-\kappa^2}\mp 2\kappa\arctan\sqrt{\dfrac{1\mp\kappa}{1\pm\kappa}}}. \tag{8}$$

Returning back to the variables (θ,p) we finally get two approximate kink solutions $(\theta_1(x,t),p_1(x,t))$ and $(\theta_2(x,t),p_2(x,t))$:

$$\theta_{1,2}=V_{1,2}+\Delta_\eta,$$
$$p_{1,2}=\frac{\partial}{\partial t}V_{1,2}-\frac{\{A\}_{-1}}{\omega}\sin(V_{1,2})-\frac{\{B\}_{-1}}{\omega}\cos(V_{1,2}), \tag{9}$$

where $V_{1,2}(x,t)=\Theta_{1,2}([x-ct]/\sqrt{1-c^2})$, $\{A\}_{-1}$ and $\{B\}_{-1}$ are defined in [1], $\Delta_\eta=\omega^2\eta/2D(1-\kappa^2)$ is a small correction that compensates the constant perturbation η far from the center of the kink. Thus, in the dissipative case the parameters of the system should satisfy the additional condition $|\Delta_\eta|\ll 1$ in order for Eq. (9) to be the good approximate kink solution.

In the particular case of sinusoidal perturbation $f(\tau)=\sin(\tau)$ one can find, using Eq. (9) of [1],

$$C=J_0(\Gamma),\quad D=-\sum_{n=1}^{\infty}\frac{J_n(\Gamma)J_{-n}(\Gamma)}{n^2},$$
$$\gamma=0,\quad \delta=\pi, \tag{10}$$

where $\Gamma=-M/\omega\sqrt{\alpha^2+\omega^2}$ (see also [2]). Note that the condition $2\gamma=\delta-\pi$ is automatically fulfilled. Using Eq. (10), one can vary the parameter β from 0 to 2π.

To conclude, we have shown that directly strongly ac driven SGE can produce two branches of kink solutions in the region of parameter space where 2π-kink localization vanishes. These two branches of kink solutions are β- and $(2\pi-\beta)$-kinks. It means that the region of parameter space where 2π-kink localization vanishes does not coincide with the regions where π-kink localization exists, to the contrary with [1].

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[1] V. Zharnitsky, I. Mitkov, and N. Gronbech-Jensen, Phys. Rev. E **58** (1), R52 (1998).

[2] K. O. Rasmussen, V. Zharnitsky, I. Mitkov, and N. Gronbech-Jensen, Phys. Rev. B **59**, 58 (1999).